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Developing Entropy of Open Finite-Level Systems

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A method to compute the time dependence of the entropy in the reduced dynamics is suggested. As a test it is applied to the Jaynes–Cummings model.

1. INTRODUCTION

Let be given an n-level quantum system described in the Hilbert space $\mathbf{C}^n$, and a quantum field described in the Hilbert space $\mathcal{H}$. Let $\mathbf{H}$ be the Hamiltonian of the composed system acting in $\mathbf{C}^n \otimes \mathcal{H}$ and ρ_0 its initial density operator. Then

$$\rho(t) = e^{-(i/\hbar)\mathbf{H}t}\, \rho_0\, e^{(i/\hbar)\mathbf{H}t}$$

and the reduced dynamics of the n-level component system is given by

$$\rho^a(t) := \mathrm{tr}_f \rho(t)$$

where the indexes a and f refer to the n-level system and the field, respectively. The partial trace tr_f is defined by the requirement

$$\mathrm{tr}(A \mathrm{tr}_f \rho(\mathbf{t})) = \mathrm{tr}((A \otimes 1)\rho(t))$$

to hold for any operator A of $\mathbf{C}^n$. In contrast to the entropy of the compound system the entropy of the component systems are time dependent. For the n-level system the entropy is

$$S^a(t) = -\mathrm{tr}\, \rho^a(t) \log \rho^a(t)$$

The calculation of $S^a(t)$ is no problem if the number of levels is less

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than or equal to four because in this case the roots of the characteristic equation are explicitly given. If the number of levels is higher, the entropy can be represented in another way which has already been sketched in ref. 1. In Section 2 this method is given in some detail. In Section 3 explicit formulas for the purities up to the second order in time are derived. In Section 4 the method is applied to the Jaynes–Cummings model.

2. THE ENTROPY AS A SERIES IN PURITIES

The entropy of a density operator of an n-level system is defined by

$$S = -\mathrm{tr}(\rho \log \rho) = -\sum_{k=1}^{n} \lambda_k \log \lambda_k$$

where the λ_k are the roots of the characteristic equation

$$\det (\lambda \mathbf{1} - \rho) = 0$$

Expanding $\log \rho$ into a power series one gets alternatively

$$S = -\mathrm{tr}(\rho \log (\mathbf{1} - (\mathbf{1} - \rho))) = \mathrm{tr} \sum_{\nu=1}^{\infty} \frac{\rho(\mathbf{1} - \rho)^{\nu}}{\nu} = \mathrm{tr} \sum_{\nu=1}^{\infty} \frac{\rho_{\nu}}{\nu}$$

where

$$\rho_{\nu} := \rho(\mathbf{1} - \rho)^{\nu}$$

Because of the characteristic equation only the powers of ρ up to the order $(n - 1)$ actually enter into the power series for S. These powers are called the *purities*

$$\mathbf{P}_k := \mathrm{tr}\, \rho^k \qquad (k = 1, 2, \ldots, n - 1)$$

This makes sense because

$$\frac{1}{n^{k-1}} \leq \mathbf{P}_k \leq 1$$

and the left-hand equality holds when the entropy is maximal and the right-hand equality holds if the density matrix expresses a pure state. In general the purities are easier to compute than the roots of the characteristic equation.

Consider now the identity

$$\det(\lambda \mathbf{1} - \rho) = \prod_{k=1}^{n} (\lambda - \lambda_k) = \sum_{j=0}^{n} (-1)^{n-j} \mu_{n-j} \lambda^j$$

where the μ_{n-j} are symmetric functions of the roots of the characteristic equation. One easily infers that ρ fulfils the *characteristic identity*

$$\sum_{j=0}^{n} (-1)^{n-j} \mu_{n-j} \rho^j = 0$$

which is true if ρ is diagonal, and, because of the unitary invariance of this equation, it holds generally. It serves to eliminate the powers higher than $(n - 1)$ from the power series for S.

Deriving expressions for the terms ρ_ν, we set

$$\sigma := \mathbf{1} - \rho$$

Then

$$\rho_\nu = \rho(\mathbf{1} - \rho)^\nu = (\mathbf{1} - \sigma)\sigma^\nu = \sigma^\nu - \sigma^{\nu+1}$$

Inserting now

$$\rho^j = \sum_{i=0}^{j} \binom{j}{i} (-1)^i \sigma^i$$

$[\binom{j}{i} = 0 \text{ for } i > j]$ into the characteristic identity, we get

$$\sum_{j=0}^{n} \sum_{i=0}^{n} \binom{j}{i} (-1)^{j+i} \mu_{n-j} \sigma^i = \sum_{i=0}^{n} \eta_{n-i} \sigma^i = 0$$

where

$$\eta_{n-i} = \sum_{j=0}^{n} \binom{j}{i} (-1)^{n-(i+j)} \mu_{n-j}$$

The desired terms ρ_ν can now be written as

$$\rho_\nu = \sigma^\nu - \sigma^{\nu+1} = \sum_{m=0}^{n-1} \beta_{n-m}^{(\nu)} \sigma^m$$

where the coefficients $\beta_{n-m}^{(\nu)}$ have to be determined.

Obviously, we have the relations

$$\beta_{n-m}^{(\nu)} = \delta_{\nu m} - \delta_{\nu(m-1)} \qquad (\nu = 1, 2, \ldots, n - 1)$$

and

$$\begin{aligned} \rho_{n-1} &= \sigma^{n-1} - \sigma^n \\ &= \sigma^{n-1} + \sum_{m=0}^{n-1} \eta_{n-m} \sigma^m \end{aligned}$$

$$= (\eta_1 + 1)\sigma^{n-1} + \sum_{m=0}^{n-2} \eta_{n-m}\, \sigma^m$$

Hence, with

$$\beta^{(\nu)} := \begin{pmatrix} \beta_1^{(\nu)} \\ \beta_2^{(\nu)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \beta_n^{(\nu)} \end{pmatrix}, \qquad \mathscr{A} := \begin{pmatrix} -\eta_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -\eta_2 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -\eta_3 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ -\eta_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ -\eta_n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}$$

we may write

$$\beta^{(1)} := \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ -1 \\ 1 \end{pmatrix}, \qquad \beta^{(\nu)} = \mathscr{A}\beta^{(\nu-1)} \qquad (\nu = 2, \ldots, n-1)$$

For $\nu = n$ we have

$$\beta^{(n)} = \mathscr{A} \begin{pmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mathscr{A}\beta^{n-1}$$

Moreover, by complete induction one proves

$$\beta^{(\nu)} = \mathscr{A}\beta^{(\nu-1)} = \mathscr{A}^{\nu-1}\beta^{(1)} \qquad (\nu \in \mathbf{N}\setminus\{1\})$$

Now the entropy is the trace of

$$\sum_{\nu=1}^{\infty} \frac{\rho_\nu}{\nu} = \sum_{\nu=1}^{\infty} \frac{1}{\nu} \sum_{m=0}^{n-1} \beta_{n-m}^{(\nu)} \sigma^n$$

$$= (\sigma^{n-1}, \sigma^{n-2}, \ldots, \sigma^0) \left(\sum_{\nu=1}^{\infty} \frac{1}{\nu} \mathscr{A}^{\nu-1} \right) \beta^{(1)}$$

Using the identity

$$\frac{1}{1 - z\mathscr{A}} = \sum_{\nu=1}^{\infty} (z\mathscr{A})^{\nu-1}$$

we get

$$\int_0^1 \frac{dz}{1 - z\mathscr{A}} = \sum_{\nu=1}^{\infty} \frac{1}{\nu} \mathscr{A}^{\nu-1}$$

Hence, the entropy can be written as

$$S = \mathrm{tr} \sum_{\nu=1}^{\infty} \frac{\rho_\nu}{\nu} = (\mathrm{tr}\sigma^{n-1}, \mathrm{tr}\sigma^{n-2}, \ldots, \mathrm{tr}\sigma^{0}) \int_0^1 \frac{dz}{1 - z\mathscr{A}} \begin{pmatrix} 0 \\ \vdots \\ -1 \\ 1 \end{pmatrix}$$

Defining the rational functions

$$f_m(z) := \left(\frac{1}{1 - z\mathscr{A}}\right)_{(n-m)n} - \left(\frac{1}{1 - z\mathscr{A}}\right)_{(n-m)(n-1)}$$

we can write the entropy in the form

$$S = \sum_{m=0}^{n-1} \mathrm{tr}\sigma^m \int_0^1 f_m(z)\, dz$$

The simple structure of $\mathscr{A}$ allows us to compute $(1 - z\mathscr{A})^{-1}$ easily: In the case of $n = 2$ we get

$$\begin{pmatrix} f_0(z) \\ f_1(z) \end{pmatrix} = \frac{1}{1 - z + \mu_2 z^2} \begin{pmatrix} \mu_2 z \\ \mu_2 \end{pmatrix}$$

For $n = 3$ we get

$$\begin{pmatrix} f_0(z) \\ f_1(z) \\ f_2(z) \end{pmatrix} = \frac{1}{1 - 2z + (\mu_2 + 1)z^2 - (\mu_2 - \mu_3)z^3} \begin{pmatrix} (\mu_3 - \mu_2)(z - z^2) \\ 1 + (\mu_2 - 1)z + (\mu_3 - \mu_2)z^2 \\ z - 1 \end{pmatrix}$$

The coefficients of the characteristic equation are bijectively related to the purities by

$$\mathbf{P}_k = \det \begin{pmatrix} 1 & 2\mu_{n-2} & 3\mu_{n-3} & \cdots & \cdots & k\mu_{n-k} \\ 1 & 1 & \mu_{n-2} & \cdots & \cdots & \mu_{n-k+1} \\ 0 & 1 & 1 & \cdots & \cdots & \mu_{n-k+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & \mu_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}$$

and

$$\mu_{n-k} = \frac{1}{k} \det \begin{pmatrix} 1 & \mathbf{P}_2 & \mathbf{P}_3 & \cdots & \cdots & \mathbf{P}_k \\ 1 & 1 & \mathbf{P}_2 & \cdots & \cdots & \mathbf{P}_{k-1} \\ 0 & 1 & 1 & \cdots & \cdots & \mathbf{P}_{k-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & \mathbf{P}_2 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}$$

3. PURITIES UP TO SECOND ORDER IN TIME

We now consider the dynamics of the compound system

$$\rho(t) = e^{-(i/\hbar)\mathbf{H}t}\, \rho_0 e^{(i/\hbar)\mathbf{H}t}$$
$$= \rho_0 + \frac{it}{\hbar}[\mathbf{H}, \rho_0] - \frac{t_2}{2\hbar^2}[\mathbf{H}, [\mathbf{H}, \rho_0]] + \ldots$$

The reduced density operator of the $\mathbf{C}^n$-component system is given, up to the second order in time, by

$$\rho^a(t) = \mathrm{tr}_f \rho_0 + t\, \mathrm{tr}_f\left(\frac{1}{i\hbar}[\mathbf{H}, \rho_0]\right) + t^2\, \mathrm{tr}_f\left(-\frac{1}{\hbar^2}[\mathbf{H}, [\mathbf{H}, \rho_0]]\right)$$
$$= \mathbf{A} + \mathbf{B}t + \mathbf{C}t^2$$

where

$$\mathbf{A} := \mathrm{tr}_f \rho_0, \qquad \mathbf{B} := \mathrm{tr}_f\left(\frac{1}{i\hbar}[\mathbf{H}, \rho_0]\right), \qquad \mathbf{C} := \mathrm{tr}_f\left(-\frac{1}{\hbar^2}[\mathbf{H}, [\mathbf{H}, \rho_0]]\right)$$

The purities are

$$\mathbf{P}_k = \mathrm{tr}_a(\rho^a(t))^k$$

and by induction one proves that

$$(\rho^a(t))^k = \mathbf{A}^k + t\sum_{l=0}^{k-1} \mathbf{A}^l\mathbf{B}\mathbf{A}^{k-l-1}$$
$$+ t^2\left(\sum_{l=0}^{k-1} \mathbf{A}^l\mathbf{C}\mathbf{A}^{k-l-1} + \sum_{m=0}^{k-2}\sum_{l=0}^{k-m-2} \mathbf{A}^l\mathbf{B}\mathbf{A}^m\mathbf{B}\mathbf{A}^{k-m-l-2}\right)$$

For uncorrelated initial states

$$\rho_0 = \rho^a \otimes \rho^f$$

The term linear in time vanishes since

$$\mathrm{tr}_a\left(\sum_{l=0}^{k-1} \mathbf{A}^l\mathbf{B}\mathbf{A}^{k-l-1}\right)$$
$$= \mathrm{tr}_a\left(\sum_{l=0}^{k-1} \mathbf{A}^{k-1}\mathbf{B} = k\mathrm{tr}_a((\mathrm{tr}_f\rho_0)^{k-1}\mathbf{B}\right)$$
$$= k\mathrm{tr}_a\left((\rho^a)^{k-1}\mathrm{tr}_f\left(\frac{1}{i\hbar}[\mathbf{H}, \rho_0]\right)\right) = k\,\mathrm{tr}\left((\rho^a)^{k-1} \otimes 1\left(\frac{1}{i\hbar}[\mathbf{H}, \rho_0]\right)\right)$$
$$= \frac{1}{i\hbar}\,\mathrm{tr}\left([\mathbf{H}, (\rho^a)^k \otimes \rho^f]\right) = 0$$

4. APPLICATION TO THE JAYNES–CUMMINGS MODEL

The Hamiltonian of the Jaynes–Cummings model is

$$H = \hbar\omega\left(1 \otimes a^+ a + \frac{1}{2}\sigma_3 \otimes 1\right) + \frac{\hbar\Omega}{2}\begin{pmatrix} 0 & a \\ a^+ & 0 \end{pmatrix}$$

We assume an uncorrelated initial state

$$\rho_0 = \begin{pmatrix} \rho_1 & \delta \\ \delta & \rho_2 \end{pmatrix} \otimes \left|\tilde{n}\right\rangle\left\langle \tilde{n}\right|$$

where $\tilde{n}$, $\tilde{n} \in \mathbf{N}_0$, is the photon number. The coefficients of $\rho^a(t)$ up to the second order are

$$\mathbf{A} = \begin{pmatrix} \rho_1 & \delta \\ \delta & \rho_2 \end{pmatrix}, \qquad \mathbf{B} = \omega \begin{pmatrix} 0 & -i\delta \\ i\delta & 0 \end{pmatrix}$$

$$\mathbf{C} = -\omega^2 \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix} + \frac{\Omega^2}{4} \begin{pmatrix} \tilde{n}(\rho_2 - \rho_1) - \rho_1 & -(\tilde{n} + \frac{1}{2})\delta \\ -(\tilde{n} + \frac{1}{2})\delta & -(\tilde{n}(\rho_2 - \rho_1) - \rho_1) \end{pmatrix}$$

Using this, we get for the purity up to the second order in time

$$\begin{aligned} \mathbf{P}_2 &= \mathrm{tr}\mathbf{A}^2 + t^2 \mathrm{tr}(\mathbf{AC} + \mathbf{CA} + \mathbf{B}^2) \\ &= \rho_1^2 + \rho_2^2 + |\delta|^2 - t^2\left(\omega^2 |\delta|^2 + \frac{\Omega^2}{2} (\tilde{n}(\rho_2 - \rho_1)^2 - \rho_1(\rho_2 - \rho_1)) \right) \end{aligned}$$

From this formula we conclude that:

1. In case $|\delta| = 0$ and $\rho_1 = \rho_2$ there is no time dependence. The purity $\mathbf{P}_2 = 1/2$ is minimal and the entropy $S = 1/2$ is maximal. This confirms that the whole system is in equilibrium when the entropy of the $\mathbf{C}^n$-component is maximal.

2. In case of $\rho_1 \neq \rho_2$ we have

$$\kappa(\rho_1) := \tilde{n}(\rho_2 - \rho_1)^2 - \rho_1(\rho_2 - \rho_1) > 0$$

This holds because

$$\kappa(\rho_1) \geq \kappa\left(\frac{1}{2} \frac{4\tilde{n} + 1}{2\tilde{n} + 1)} \right) = 2\tilde{n} \frac{2\tilde{n}^2 + 4\tilde{n} + 1}{(2\tilde{n} + 1)^2} \geq 0$$

Hence, since $\tilde{n} \geq 0$, the assumption $\kappa(\rho_1) = 0$ implies $\tilde{n} = 0$ and, finally, $\rho_1 = \rho_2 = 1/2$, in contradiction to the hypothesis. In conclusion the coefficient of t_2 is negative s.t.

$$\frac{d}{dt} \mathbf{P}_2 = 0 \qquad \text{and} \qquad \frac{d^2}{dt^2} \mathbf{P}_2 \leq 0$$

If $\rho_1 \neq \rho_2$, the purity begins to decrease parabolically. Since the state becomes entangled with the field state because of the interaction the present discussion does not show that the purity decreases monotonically until the minimum is reached.

Clearly, these conclusions are not much. The purpose was only to illustrate the proposed method on a simple model. The Jaynes–Cummings model is integrable and the closed form of the solution is well known. This has been used in ref. 2, where in contrast to the above consideration the field is considered as the open system and the dynamics of classical-like entropies have been discussed. Our method does not assume that a closed solution is known.

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